Stochastic Models on a Ring and Quadratic Algebras.
The Three Species Diffusion Problem.

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The stationary state of a stochastic process on a ring can be expressed using traces of monomials of an associative algebra defined by quadratic relations. If one considers only exclusion processes one can restrict the type of algebras and obtain recurrence relations for the traces. This is possible only if the rates satisfy certain compatibility conditions. These conditions are derived and the recurrence relations solved giving representations of the algebras.

1. Introduction

In the preceding paper [1] we have considered the application of quadratic algebras to stochastic problems with closed or open boundaries, here we study the case of periodic boundary conditions. Again, we are going to be interested in the (unnormalized) probability distributions describing stationary states. In the language of quantum chains, we are looking for ground states which have momentum and energy zero. A lot of work was already done looking for matrix-product states in the case of periodic chains [2, 3, 4]; the present approach is different in the sense that it is based on the existence of recurrence relations which can be solved using representations of some quadratic algebras. The idea of this approach is not new [5, 6] and all we did is to pursue it in a consistent way in the case of models with three states. As a result one finds more solutions than were known before which can be used either to repeat the previous applications [5, 6] in a more general framework or to look for novel applications. The same approach can be extended to problems with more states. Such an extension is not trivial.

From a mathematical point of view one has to solve a well stated problem: given a certain class of quadratic algebras one has to find those which are compatible with the trace operation. One lesson one learns from the present work is that, unexpectedly, in order to solve the periodic case one makes use of Fock representations, derived in the previous paper [1], which were necessary to solve the problem with closed or open boundaries.

We first consider the general case of N species on a ring with L sites and use the notations of Ref.[1]. On each site we take a stochastic variable β_k ($\beta = 0, 1, \ldots, N-1$ and $k = 1, 2, \ldots, L$), on each link k between the sites k and $k+1$ the rates $\Gamma^{\gamma_k \gamma_{k+1}}_{\beta_k \beta_{k+1}}$ give the probability per unit time for the transition

$$\{\ldots, \gamma_k, \gamma_{k+1}, \ldots\} \mapsto \{\ldots, \beta_k, \beta_{k+1}, \ldots\} \ . \tag{1.1}$$

The Hamiltonian associated with the master equation is [1]:

$$H = -\sum_{k=1}^{L} \Gamma^{\alpha\beta}_{\gamma\delta} \, E^{\gamma\alpha}_k \, E^{\delta\beta}_{k+1} \tag{1.2}$$

where the matrices E_k act on the kth site and have matrix elements

$$\left(E^{\alpha\beta} \right)_{\gamma\delta} = \delta_{\alpha\gamma} \delta_{\beta\delta} \ , \tag{1.3}$$

and the diagonal elements $\Gamma^{\alpha\beta}_{\alpha\beta}$ are given by

$$\sum_{(\gamma,\delta)} \Gamma^{\alpha\beta}_{\gamma\delta} = 0 \ . \tag{1.4}$$

The site $L+1$ is identified with the first site.

Now it is trivial to show that if we take N matrices D_α ($\alpha = 0, 1, \ldots, N-1$) and N matrices X_α satisfying the quadratic algebra

$$\sum_{\alpha,\beta=0}^{N-1} \Gamma^{\alpha\beta}_{\gamma\delta} \, D_\alpha \, D_\beta = X_\gamma \, D_\delta - D_\gamma \, X_\delta \qquad (\gamma, \delta = 0, 1, \ldots, N-1) \tag{1.5}$$

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then

$$P_s = \text{Tr} \left(\prod_{k=1}^{L} \left(\sum_{\alpha=0}^{N-1} D_\alpha u_\alpha^{(k)} \right) \right) \tag{1.6}$$

is a stationary state:

$$H \cdot P_s = 0 \tag{1.7}$$

We have denoted by $u_\alpha^{(k)}$ ($\alpha = 0, 1, \ldots, N-1$ and $k = 1, 2, \ldots, L$) the basis vectors in the vector space associated to the kth site on which the basis matrices $E_k^{\alpha,\beta}$ of Eq.(1.2) act. The trace operation in Eq.(1.6) is taken in the auxiliary space of the D_α and X_α matrices. We notice that the bulk algebra (1.5) is identical to the one encountered in the previous paper. The new thing is the appearance of the trace operation in Eq.(1.6).

As opposed to the problem with closed and open boundaries where the bulk algebra was completed by the condition of the existence of a Fock representation defined by the boundary conditions and where it was shown that D and X matrices can be derived once the bulk and boundary rates are given [7], in the present case very little is known, except that the bulk algebra exists since a representation for the D's and X's is known [7]. This representation however is pathologic in that the traces of any monomial of D's vanish. It is also not clear if all stationary states can be obtained through the ansatz (1.5) [8].

The remarkable thing about the algebras (1.5) is that, if the ground state (1.6) is unique, all the traces of monomials of degree L and containing only D_α's and no X_α's are, up to a common factor, independent of the representation of the algebra which implies that in order to compute them one can take the smallest one.

Last but not least, let us observe that the cases (see for example [4]) where the ansatz was applied correspond to representations with $X_\alpha = 0$ [8]. This leads us to polynomial algebras like in Sec.3 of Ref.[1].

We will now restrict our problem looking only to simple exclusion processes. This means that only the rates $\Gamma_{\beta\alpha}^{\alpha\beta} = g_{\alpha\beta}$ and the diagonal ones are non-zero. Also, we will look for solutions in which the X_α matrices are c-numbers x_α. This last assumption will imply conditions on the rates $g_{\alpha\beta}$. The quadratic algebras have now a simple form:

$$g_{\alpha\beta} D_\alpha D_\beta - g_{\beta\alpha} D_\beta D_\alpha = x_\beta D_\alpha - x_\alpha D_\beta \qquad (\alpha, \beta = 0, 1, \ldots, N-1) \tag{1.8}$$

There are $N(N-1)/2$ relations with N parameters x_α and N generators D_α.

The appearance of N arbitrary parameters in the algebra can be understood in the following way. The problem has a $U(1)^{N-1}$ symmetry corresponding to the conservation of the number of particles of $N-1$ species (the remaining species are the vacancies). The ground state is highly degenerate. If one has a ring with L sites, P_s given by Eq.(1.6) has the following formal expression:

$$P_s = \sum_{n_\alpha} d_0{}^{n_0} d_1{}^{n_1} \cdots d_{N-1}{}^{n_{N-1}} A_{n_0, n_1, \ldots, n_{N-1}} \tag{1.9}$$

where

$$\sum_{\alpha=0}^{N-1} n_\alpha = L \tag{1.10}$$

The d_α are arbitrary parameters and $A_{n_0, n_1, \ldots, n_{N-1}}$ are vectors on which H acts. In this way, in each sector, given by the number n_0 of vacancies and n_i of particles of type i, the ground state can be identified. That P_s is indeed of the form (1.9) can be seen from the invariance of the algebra (1.8) under the transformation:

$$D_\alpha \mapsto d_\alpha D_\alpha, \qquad x_\alpha \mapsto d_\alpha x_\alpha \tag{1.11}$$

Let us stress again that in order to obtain the expression (1.9) one can take any representation of the algebra. We now turn our attention to the algebra (1.8).

What we need is not only to have associative algebras but that the trace operation exists. Since the left hand side of the Eqs.(1.8) is quadratic and the right hand side linear in the generators D_α, this implies recurrence relations among traces of monomials and hence compatibility relations among the rates $g_{\alpha\beta}$. In order to solve this problem, our strategy was the following: we have first considered monomials of a low degree (up to five), got the compatibility relations and made sure that no new relations occur from higher degree monomials by finding a representation with finite traces for the algebra. Once we have the algebra, for the physical applications one can do calculations using either directly the algebra and do formal manipulations under the trace operation or use the representation. This procedure is going to be explained in detail in section 3 for the three-state case. In the short section 2 we ask a simple question: what are the conditions on the $g_{\alpha\beta}$ for arbitrary N such that one has one-dimensional representations? (one-dimensional representations obviously have a trace.) Finally, in section 4 we present our conclusions.

2. One-dimensional representations

The simplest examples of the algebras defined by Eq.(1.8) which have a trace are those in which one has one-dimensional representations. In order to find them, we take $D_\alpha = d_\alpha$, arbitrary non-zero c-numbers. It is convenient to introduce the notations:

$$a_{\alpha\beta} = g_{\alpha\beta} - g_{\beta\alpha} \tag{2.1}$$

Using Eq.(1.8) one obtains:

$$\frac{x_\alpha}{d_\alpha} - \frac{x_\beta}{d_\beta} = a_{\beta\alpha} \qquad (\alpha, \beta = 0, 1, \ldots, N-1) \tag{2.2}$$

These equations determine the parameters x_α once the d_α are chosen. One obtains $(N-1)(N-2)/2$ conditions on the rates:

$$a_{0\alpha} - a_{0\beta} = a_{\beta\alpha} \qquad (\alpha, \beta = 1, 2, \ldots, N-1) \tag{2.3}$$

and the parameters x_α are

$$x_\alpha = d_\alpha(a_{0\alpha} + x_0/d_0) \qquad (\alpha = 1, 2, \ldots, N-1) \tag{2.4}$$

Notice that one of them can be chosen at will. In Eqs.(2.3) and (2.4) we have singled out $\alpha = 0$ as a matter of notational convenience.

The wave function, see Eq.(1.6), is symmetric. This observation is interesting for the following reason. In the case of simple exclusion processes, the Hamiltonian given by Eq.(1.2) has a $U(1)^{N-1}$ symmetry. As already discussed, this corresponds to the conservation of the $N-1$ types of particles hopping among vacancies. The symmetric wave function, however, corresponds to an $SU(N)$ representation given by the Young tableau with one row and L boxes (L is the number of sites), although the Hamiltonian does not have this symmetry. From Eq.(2.3) we also see that for $N = 2$ one has one-dimensional representations for any rates.

3. The three-state algebras

We will now study in detail the algebras given by Eq.(1.8) for $N = 3$. In order to find them, we will consider several cases.

3.1. x_0, x_1 and x_2 different from zero

We define

$$D_i = x_i E_i \qquad (i = 1, 2, 3) \tag{3.1}$$

and get the algebra:

$$\begin{aligned}
g_{01} E_0 E_1 - g_{10} E_1 E_0 &= E_0 - E_1 \\
g_{20} E_2 E_0 - g_{02} E_0 E_2 &= E_2 - E_0 \\
g_{12} E_1 E_2 - g_{21} E_2 E_1 &= E_1 - E_2
\end{aligned} \tag{3.2}$$

From writing the recurrence relations for monomials of degree two and three, the equations giving $\mathrm{Tr}(E_0 E_1 E_2)$ and $\mathrm{Tr}(E_2 E_1 E_0)$ are consistent only if

$$a_{01} - a_{02} = a_{21} \tag{3.3}$$

or if

$$\begin{aligned}
a_{12} (g_{10} g_{20} - g_{01} g_{02}) \, \mathrm{Tr}(E_0) + \\
a_{20} (g_{01} g_{21} - g_{10} g_{12}) \, \mathrm{Tr}(E_1) + \\
a_{01} (g_{02} g_{12} - g_{20} g_{21}) \, \mathrm{Tr}(E_2) = 0
\end{aligned} \tag{3.4}$$

Eq.(3.3) gives the condition to have a one-dimensional representation, see (2.3). Eq.(3.4) however is new. We use (3.4) to express $\mathrm{Tr}(E_0)$ in terms of $\mathrm{Tr}(E_1)$ and $\mathrm{Tr}(E_2)$ and look for monomials of degree four. No new conditions appear. (This implies that the ground-states for chains of up to four sites can be obtained by this method.) For monomials of degree five, however, the consistency conditions give that the traces of all monomials of degree two to four are zero. Since we went up to monomials of degree five one can guess what kind of dirty algebra one had to do (also [9]). We looked without success for conditions on the rates in order to find non-zero solutions. The equations are however so cumbersome that we can't even be sure that we didn't miss one.

We have also looked for finite-dimensional representations of the algebra also with a negative result. As a by-product we have found that the algebra (3.2 exists if

$$g_{10}g_{01} = g_{20}g_{02}, \quad g_{12} = -g_{21} = \frac{g_{20}(g_{02} - g_{01})}{g_{01} + g_{20}} \tag{3.5}$$

Notice that this condition is incompatible with positivity of the rates. This algebra has a two-dimensional representation:

$$\mathcal{E}_0 = \frac{1}{g_{20} - g_{01}} \begin{pmatrix} g_{01}/g_{02} & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathcal{E}_1 = \frac{1}{g_{01} - g_{02}} \begin{pmatrix} 1 & 0 \\ -(g_{01}{}^2 + g_{20}{}^2)/\lambda g_{20}{}^2 & g_{01}/g_{20} \end{pmatrix} \tag{3.6}$$

$$\mathcal{E}_2 = \frac{1}{g_{02} - g_{01}} \begin{pmatrix} g_{01}/g_{20} & \lambda \\ 0 & 1 \end{pmatrix}.$$

Here λ is an arbitrary parameter.

3.2. $x_0 = 0$, x_1 and x_2 different from zero

We define

$$D_1 = x_1 E_1, \qquad D_2 = x_2 E_2 \tag{3.7}$$

and the algebra (1.8) becomes:

$$g_{01} D_0 E_1 - g_{10} E_1 D_0 = D_0$$
$$g_{02} D_0 E_2 - g_{20} E_2 D_0 = D_0 \tag{3.8}$$
$$g_{12} E_1 E_2 - g_{21} E_2 E_1 = E_1 - E_2$$

This algebra has a special structure in the sense that all the independent monomials in D_0, E_0 and E_2 can be organized in the following way:

$$P_0, \quad D_0 P_1, \quad D_0{}^2 P_2, \dots \tag{3.9}$$

where the P_i are monomials in E_1 and E_2 only. This will imply in the trace problem a decoupling of the recurrence relations according to the power of D_0 appearing in the monomials. In particular for words without D_0's, we can take $D_0 = 0$ in Eq.(3.8) and are left with the $N = 2$ algebra containing E_1 and E_2 for which we know that we have one-dimensional representations and thus in this sector the problem is solved. The problem is of course marrying the last equation in (3.8) with the first two. The decoupling of the trace problem in various sectors will have also an unexpected consequence in the representations of the algebra. The representations with a trace for words containing D_0's will not have a trace for words not containing any D_0. So much for the structure of the algebra (3.8).

Going up to monomials of order three, the consistency relations obtained from the equations giving $\mathrm{Tr}(D_0 E_1 E_2)$ and $\mathrm{Tr}(E_2 E_1 D_0)$ give again Eq.(3.3) (this is compatible with Eq.(2.4) in which one can take $x_0 = 0$) or

$$g_{01} g_{02} = g_{10} g_{20} \ . \tag{3.10}$$

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We first assume

$$g_{01}\,,\ g_{20}\,,\ g_{10}\,,\ g_{02} \neq 0 \tag{3.11}$$

and look at words of order four. One gets two new conditions which together with (3.10) give:

$$g_{10} = g_{02}\,, \qquad g_{01} = g_{20}\,, \qquad g_{21} - g_{12} = g_{01} - g_{10} \tag{3.12}$$

We will now introduce the following notations:

$$q = \frac{g_{01}}{g_{10}} = \frac{g_{20}}{g_{02}}\,, \qquad r = \frac{g_{21}}{g_{12}} \tag{3.13}$$

and

$$E_1 = \frac{G_1}{g_{01} - g_{10}}\,; \ E_2 = \frac{G_2}{g_{10} - g_{01}}\ . \tag{3.14}$$

Taking into account the conditions (3.11) on the rates, the new algebra is:

$$
\begin{aligned}
q\,D_0\,G_1 - G_1\,D_0 &= (q-1)\,D_0 \\
q\,G_2\,D_0 - D_0\,G_2 &= (q-1)\,D_0 \\
r\,G_2\,G_1 - G_1\,G_2 &= (r-1)\,(G_1 + G_2)
\end{aligned}
\tag{3.15}
$$

At this point we are not going to look at words of order five or more but show that a representation with a trace exits. Before we show this, let us first notice that the algebra (3.15) is invariant under the transformation:

$$D_0 \mapsto D_0\,,\ G_1 \mapsto G_2\,,\ G_2 \mapsto G_1\,,\ q \mapsto \frac{1}{q}\,,\ r \mapsto \frac{1}{r} \tag{3.16}$$

and that one has the identities:

$$
\begin{aligned}
q^n\,D_0{}^n\,G_1 - G_1\,D_0{}^n &= (q^n - 1)\,D_0{}^n \\
q^n\,G_2\,D_0{}^n - D_0{}^n\,G_2 &= (q^n - 1)\,D_0{}^n
\end{aligned}
\tag{3.17}
$$

The Eqs.(3.16) and (3.17) allow to find traces on some monomials if one knows some others.

In order to show that a representation exists, we first write

$$
\begin{aligned}
G_1 &= 1 - \sqrt{r-1}\,\mathcal{A} \\
G_2 &= 1 + \sqrt{r-1}\,\mathcal{B} \\
D_0 &= d_0(1 + (q-1)\,\mathcal{N})
\end{aligned}
\tag{3.18}
$$

where d_0 is an arbitrary parameter. Using (3.15) we get:

$$
\begin{aligned}
\mathcal{A}\,\mathcal{B} - r\,\mathcal{B}\,\mathcal{A} &= 1 \\
\mathcal{A}\,\mathcal{N} - q\,\mathcal{N}\,\mathcal{A} &= \mathcal{A} \\
\mathcal{N}\,\mathcal{B} - q\,\mathcal{B}\,\mathcal{N} &= \mathcal{B}
\end{aligned}
\tag{3.19}
$$

The algebra (3.19) which contains an r–deformed harmonic oscillator (generators $\mathcal{A}$ and $\mathcal{B}$) together with q–deformed actions of the number operator $\mathcal{N}$ has a Fock representation [1, 10]:

$$\mathcal{A}\,|0> = <0|\,\mathcal{B} = 0 \,, \qquad \mathcal{B} = \mathcal{A}^T \tag{3.20}$$

where $\mathcal{A}^T$ is the transpose of $\mathcal{A}$,

$$\mathcal{A} = \begin{pmatrix} 0 & g_1 & 0 & 0 & \cdots \\ 0 & 0 & g_2 & 0 & \\ 0 & 0 & 0 & g_3 & \\ \vdots & & & \ddots & \ddots \end{pmatrix}, \qquad \mathcal{N} = \begin{pmatrix} p_1 & 0 & 0 & \cdots \\ 0 & p_2 & 0 & \\ 0 & 0 & p_3 & \\ \vdots & & & \ddots \end{pmatrix} \tag{3.21}$$

and

$$g_n{}^2 = \{n\}_r, \qquad p_n = \{n-1\}_q, \qquad \{n\}_\lambda = \frac{\lambda^n - 1}{\lambda - 1} \tag{3.22}$$

It is convenient to denote:

$$G_1 = 1 + \mathcal{F}_1, \qquad G_2 = 1 + \mathcal{F}_2, \qquad D_0 = d_0\,\mathcal{I}(q) \tag{3.23}$$

The matrices $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{I}(q)$ have a simple form:

$$\mathcal{F}_1 = \begin{pmatrix} 0 & -f_1 & 0 & 0 & \cdots \\ 0 & 0 & -f_2 & 0 & \\ 0 & 0 & 0 & -f_3 & \\ \vdots & & & \ddots & \ddots \end{pmatrix}, \quad \mathcal{F}_2 = -\mathcal{F}_1^T, \quad \mathcal{I}(q) = \begin{pmatrix} e_1 & 0 & 0 & \cdots \\ 0 & e_2 & 0 & \\ 0 & 0 & e_3 & \\ \vdots & & & \ddots \end{pmatrix} \tag{3.24}$$

where

$$f_k{}^2 = r^k - 1, \qquad e_k = q^{k-1} \tag{3.25}$$

Let us now notice the following useful relations:

$$\begin{aligned} q\,\mathcal{I}(q)\,\mathcal{F}_1 &= \mathcal{F}_1\,\mathcal{I}(q) \\ q\,\mathcal{F}_2\,\mathcal{I}(q) &= \mathcal{I}(q)\,\mathcal{F}_2 \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} \mathcal{F}_1\,\mathcal{F}_2 &= 1 - r\,\mathcal{I}(r) \\ \mathcal{F}_2\,\mathcal{F}_1 &= 1 - \mathcal{I}(r) \end{aligned} \tag{3.27}$$

as well as

$$\mathcal{I}(r)\,\mathcal{I}(q) = \mathcal{I}(rq) \tag{3.28}$$

and

$$\mathrm{Tr}\,\mathcal{I}(\lambda) = \frac{1}{1 - \lambda} \tag{3.29}$$

The calculation of the trace of any monomial containing at least one D_0 proceeds as follows. Using Eq.(3.24) one has to compute only traces containing an equal number

of $\mathcal{F}_1$'s and $\mathcal{F}_2$'s together with $\mathcal{I}(q)$'s. We use the commutation relations (3.26) to bring the $\mathcal{I}(q)$'s together and "condense" then into one using Eq.(3.28). Next we use Eq.(3.27) in order to express products of $\mathcal{F}_1$'s and $\mathcal{F}_2$'s in terms of $\mathcal{I}(r)$'s and make them "condense" together with the $\mathcal{I}(q^n)$ obtained from, let us say, n $\mathcal{I}(q)$'s. The final result is an expression which contains $\mathcal{I}$'s of various arguments each one having a trace given by Eq.(3.29). The d_i of Eq.(1.9) are d_0, $d_1 = x_1$ and $d_2 = x_2$. This concludes our discussion of the algebra with the conditions (3.12). The special case $q = r$ is already known and was applied in Ref.[5].

We now return to Eq.(3.10) and consider the case

$$g_{01} = g_{20} = 0 \tag{3.30}$$

We can find directly representations of the algebra in this case.

First we consider

$$\mu \equiv \frac{g_{10}}{g_{21} - g_{12}} \neq 1, \quad \nu \equiv \frac{g_{02}}{g_{21} - g_{12}} \neq 1 . \tag{3.31}$$

We make a change of notations:

$$E_1 = -\frac{G_1}{g_{10}}, \qquad E_2 = \frac{G_2}{g_{02}} \tag{3.32}$$

and instead of the algebra (3.8) we get:

$$\begin{aligned}
G_1 \, D_0 &= D_0 \\
D_0 \, G_2 &= D_0 \\
g_{12} \, G_1 \, G_2 - g_{21} \, G_2 \, G_1 &= g_{02} \, G_1 + g_{10} \, G_2
\end{aligned} \tag{3.33}$$

Similar to what we did in the case of the last algebra (see Eq.(3.23)), we write:

$$G_1 = 1 + \mathcal{F}_1, \qquad G_2 = 1 + \mathcal{F}_2, \qquad D_0 = d_0 \, \mathcal{F}_0 \tag{3.34}$$

and make the observation that the Fock representation of the following algebra [1, 10]:

$$\begin{aligned}
\mathcal{A} \, \mathcal{I}(0) &= 0 \\
\mathcal{I}(0) \, \mathcal{B} &= 0 \\
\xi(\mathcal{A} \, \mathcal{B} - r \, \mathcal{B} \, \mathcal{A}) &= \mathcal{A} + \mathcal{B} + 1 \\
\mathcal{A} \, |0> &=< 0| \, \mathcal{B} = 0 \\
\mathcal{B} &= \mathcal{A}^T
\end{aligned} \tag{3.35}$$

is known:

$$\mathcal{A} = \begin{pmatrix} a_1 & k_1 & 0 & 0 & \cdots \\ 0 & a_2 & k_2 & 0 & \\ 0 & 0 & a_3 & k_3 & \\ \vdots & & & \ddots & \ddots \end{pmatrix} \tag{3.36}$$

where

$$a_n = \frac{1}{\xi}\frac{r^{n-1}-1}{r-1}, \qquad k_n{}^2 = \frac{1}{\xi^2}\frac{r^n-1}{r-1}\left(\xi + \frac{r^{n-1}-1}{r-1}\right) \quad, \tag{3.37}$$

and $\mathcal{I}(q=0)$ as in (3.24). We introduce the following notations for ratios of rates

$$r = \frac{g_{21}}{g_{12}}, \quad \xi = \frac{\mu+\nu-1}{(\mu-1)(\nu-1)} \quad. \tag{3.38}$$

Using (3.35)–(3.37) we find a representation for $\mathcal{F}_0$, $\mathcal{F}_1$ and $\mathcal{F}_2$ of Eq.(3.34):

$$\begin{aligned}
\mathcal{F}_0 &= \mathcal{I}(0)\\
\mathcal{F}_1 &= (\mu-1)(\mathcal{I}(r)+\mathcal{V}-1)\\
\mathcal{F}_2 &= (\nu-1)(\mathcal{I}(r)+\mathcal{V}^T-1)
\end{aligned} \tag{3.39}$$

where $\mathcal{I}(r)$ is defined as above and

$$\mathcal{V} = \begin{pmatrix}
0 & s_1 & 0 & 0 & \cdots \\
0 & 0 & s_2 & 0 & \\
0 & 0 & 0 & s_3 & \\
\vdots & & & \ddots & \ddots
\end{pmatrix} \tag{3.40}$$

with

$$s_n{}^2 = (r^n-1)(\xi + r^{n-1}-1) \quad. \tag{3.41}$$

Notice that

$$r\,\mathcal{I}(r)\,\mathcal{V} = \mathcal{V}\,\mathcal{I}(r) \quad, \tag{3.42}$$

and that the products $\mathcal{V}\mathcal{V}^T$ and $\mathcal{V}^T\mathcal{V}$ can be written in terms of $\mathcal{I}(r)$ and $\mathcal{I}(r^2)$. This makes the discussion of the existence and calculations of the traces to be identical to that of the previous algebra. A special case of this algebra when $r = 0$ was already discussed in Ref.[6].

For the case $\mu = 1$ one must use a different representation (the case ν is similar). We write

$$E_1 = \frac{1}{g_{10}}\left(1 + \frac{g_{02}}{g_{02}-g_{10}}\mathcal{A}\right) \qquad E_2 = -\frac{1}{g_{02}}\left(1 + \frac{g_{02}-g_{10}}{g_{12}}\mathcal{B}\right) \qquad D_0 = d_0\,\mathcal{I}(0) \quad (3.43)$$

where now $\mathcal{A}$ and $\mathcal{B}$ fulfill:

$$\begin{aligned}
\mathcal{A}\,\mathcal{I}(0) &= 0\\
\mathcal{I}(0)\,\mathcal{B} &= 0\\
\mathcal{A}\,\mathcal{B} - r\,\mathcal{B}\,\mathcal{A} &= \mathcal{A} + 1 \tag{3.44}\\
\mathcal{A}\,|0> &=< 0|\,\mathcal{B} = 0 \tag{3.45}
\end{aligned}$$

The matrices are given by

$$
\mathcal{A} = \begin{pmatrix} 0 & f_1 & 0 & 0 & \cdots \\ 0 & 0 & f_2 & 0 & \\ 0 & 0 & 0 & f_3 & \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & & & & \ddots \end{pmatrix}, \quad
\mathcal{B} = \begin{pmatrix} b_1 & 0 & 0 & 0 & \cdots \\ f_1 & b_2 & 0 & 0 & \\ 0 & f_2 & b_3 & 0 & \\ 0 & 0 & f_3 & b_4 & \\ \vdots & & & \ddots & \ddots \end{pmatrix}
\tag{3.46}
$$

with

$$
b_n = \frac{r^{n-1} - 1}{r - 1}, \qquad f_n{}^2 = \frac{r^n - 1}{r - 1}
\tag{3.47}
$$

Compare with Eqs.(3.35)–(3.37).

The profound reason which explains why the same representations for the E_α occur in the problem with periodic boundary conditions and for the open end problems (see Ref.[1], Sec.7) comes from the following observation. The differences between the two types of boundary conditions is that the parameters x_1 and x_2 are free for the periodic case but are determined by the boundary matrices in the open case. In the latter case supplementary relations exist among the bulk $(g_{\alpha\beta})$ and boundary rates. Since the E_α found in the case of open ends have a trace, they can be used in the present paper.

3.3. $x_0 = x_1 = 0$, x_2 different from zero

In this case the algebra (1.8) becomes:

$$
\begin{aligned}
g_{02}\, D_0\, E_2 - g_{20}\, E_2\, D_0 &= D_0 \\
g_{12}\, D_1\, E_2 - g_{21}\, E_2\, D_1 &= D_1 \\
g_{01}\, D_0\, D_1 - g_{10}\, D_1\, D_0 &= 0
\end{aligned}
\tag{3.48}
$$

where we have taken $D_2 = x_2\, E_2$. If one wants non-trivial ground states with particles of the three species, one has to take:

$$
g_{01} = g_{10} = 0
\tag{3.49}
$$

In this case the symmetry of the problem is enormous since no prescription is given on products of D_0's and D_1's. The right language are affine symmetries [11] and it is beyond the scope of this paper to get involved in the representation theory for this case. The algebra with traces however very probably exists since it was used already in Ref. [12] in the sector of one D_0 and one D_1 and there are no obvious reasons why it should not be so in the general case. This model was also investigated in [13] by different methods.

Finally the case $x_0 = x_1 = x_2 = 0$ takes us the situation of symmetric rates where, as we already know, we get a symmetric wave function.

4. Conclusion

Leaving aside physical applications, our own fascination with the problem described in this paper comes from the unusual properties of the representations of the algebra appearing in searching for matrix-products ground-states. As stressed already in the introduction certain monomials or even all monomials of the same degree in D's have, up to a normalisation factor, traces independent of the representation. More has to be understood in the general case when the X's are matrices (see Eq.(1.5)) and also in the class of algebras given by Eq.(1.8). It is probably possible to encode the conditions on the rates $g_{\alpha\beta}$ in some cubic and quartic identities. This guess is based on the fact that conditions found on the $g_{\alpha\beta}$ were obtained from words of degree three and four. For this reason the four-state problem is worth looking at in order to see if a general pattern appears. From the point of view of physical applications, for the three-state problem we have solved the recurrence relations if the rates satisfy the conditions (3.3), (3.12) or (3.30). The case given by Eq.(3.49) was known already. No other cases exist. However, as everybody knows from solving the recurrence relations to computing a relevant physical quantity it is a long way to go.

Acknowledgments

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